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On Fröhlich's solution for Boussinesq's problem

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SUMMARY

The *concentration factor* introduced by O. K. Fröhlich is visualized as a procedure for examining the pattern of load transfer from surface loads to the interior of a geomaterial. The historical details that led to the introduction of the concentration factor are scant although it is widely used in the area of soil mechanics problems associated with tillage-induced soil compaction. The purpose of this note is to examine the concentration factor in terms of the geomechanics of an elastic continuum and to identify the precise conditions that are satisfied by the distribution of stresses and strains that accommodate the concentration factor.

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1. INTRODUCTION

Otto Karl Fröhlich (1885–1964) studied engineering at the Technical University of Graz and obtained his doctoral degree in engineering in 1911. He worked in Berlin, St. Petersburg, Amsterdam and in The Hague from 1934–1935. His book, entitled, '*Druckverteilung im Baugrunde. Mit Besonderer Berücksichtigung der Plastischen Erscheinungen*' [1] was published in 1934 during his stay in Gravenhage, the Netherlands and dedicated to his teacher A. Föppl. In 1935, he was invited by Karl Terzaghi to take up the position of lecturer and was appointed Professor of Soil Mechanics and Geotechnical Engineering at the Technical University of Vienna in 1940. During his tenure at the Technical University of Vienna, he collaborated extensively with Terzaghi and, in 1936, they published '*Theorie der Setzung von Tonschichten*' [2] dealing with consolidation of clay layers. An extensive account of the disputes between Terzaghi and Fillunger [3] during the publication of this work is also given by de Boer [4].

The problem of load transfer from a concentrated normal force to an isotropic halfspace region was first presented by Boussinesq [5] and represents a classical result that is widely used in geomechanics and applied mechanics [6, 7]. Boussinesq's approach for solving the problem of the loading of a halfspace by a concentrated normal force (Figure 1) takes into account all the equations governing the classical theory of elasticity and the relevant boundary conditions and regularity conditions. The solution is obtained by appeal to results of potential theory and yields an exact closed form solution for the displacements and stresses within the halfspace region. An alternative approach to the development of the problem was presented by Selvadurai [8, 9], and the classical result by Mindlin [10] represents the generalized result from which both Boussinesq's solution for the problem of the normal loading of the surface of a halfspace region

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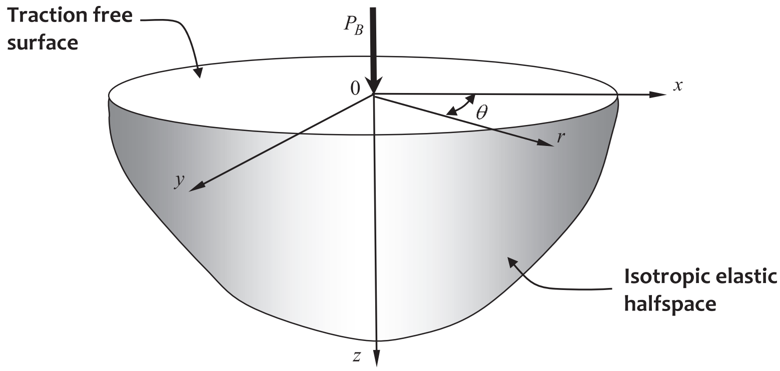

Figure 1. Boussinesq's problem for an isotropic elastic half space.

by a concentrated force and Kelvin's solution [11–13] for the interior loading of an isotropic elastic halfspace by a concentrated force can be recovered as special cases. Studies dealing with this class of fundamental solutions are quite extensive, and the applications have been extended to include both anisotropy and heterogeneity of the elastic medium [14–17].

The concept of a 'concentration factor' (n) for the study of the pattern of load transfer to the interior of an isotropic elastic halfspace region appears to have been first proposed by Griffith [18] (see also Soehne [19] and Dexter *et al*. [20]) and introduced and documented in the book by Fröhlich [1]. The ability to alter the distribution of the load transfer pattern in the interior of the halfspace region through the alteration of a single coefficient was a major attraction of the theory, and for the specific value of the 'concentration factor', $n = 3$, Boussinesq's solution is recovered. The physical basis for the 'concentration factor' was, however, not provided by Fröhlich [1]. Ohde [21] and Borowicka [22] indicate that in addition to the limit of $n = 3$, the case $n = 4$ corresponds to an elastic halfspace region where the linear elastic shear modulus varies linearly with depth. The result of Borowicka's study [22] involves an infinite series solution in terms of the Poisson number. The complete solution to the problem of the surface loading of an isotropic elastic halfspace with a linear variation of the linear elastic shear modulus was first solved by Gibson [23] (see also [7, 24–27]). Ohde [21] arrives at a relationship between the 'concentration factor' n and Poisson's ratio ν in the form $n = (1 + \nu^{-1})$, giving the result $n = 3$, which is required for the solution to reduce to Boussinesq's classical result for the special case of an incompressible elastic material. Returning to Fröhlich's result, Terzaghi [28], Tschebotarioff [29], Jumikis [30] and Széchy and Varga [31] present arguments that also follow those presented by Ohde [21] with regard to the depth-dependent variation of the elastic moduli. Application of the concept of depth-dependent nonhomogeneity and its connection to Fröhlich's concentration factor is also presented by Klein [32] (see also Koronev [33]).

The resurgence in the interest in the application of Fröhlich's concentration factor is largely due to the potential applicability of the result to explain the deviations in the stress transfer at depth due to the effects of soil compaction. Measurements of stress distribution within soil masses during the application of surface loads have been documented by several investigators, and a recent review with applications to mechanics of soil tillage is given by Keller *et al*. [34]. At the outset, it should be remarked that the measurement of stresses within soil masses using embedded contact pressure cells is a difficult exercise, largely due to the fact that the cell-action factor that is needed to correctly interpret the stress state will be governed by a variety of responses including the constitutive relationship for the soil itself. Extensive reviews of contact stress measurement and the development of techniques for the interpretation of results from soil pressure cells are given by Hast [35], Hvorslev [36], Selvadurai [37], Hanna [38] and Selvadurai *et al*. [39]. The results derived from embedded pressure cells are considerably more difficult to interpret than data from pressure cells that are located at a rigid boundary. For this reason, the experimental results themselves can be prone to incorrect interpretation. Regardless of this limitation, the result of Fröhlich's study [1] is extensively used in current approaches to examining compaction-induced alterations of the soil

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fabric and its load transfer capabilities. The theoretical developments of Fröhlich therefore merit further discussion so that its basis can be examined in the context of theoretical geomechanics.

2. BOUSSINESQ'S PROBLEM

We consider the problem of the concentrated normal force P_B that is applied at the surface of a semi-infinite medium (Figure 1). In the case where the semi-infinite solid is an isotropic elastic continuum, this corresponds to Boussinesq's classical problem that can be developed using a variety of approaches. These are discussed in several recent articles (e.g. Timoshenko and Goodier [40], Little [41], Davis and Selvadurai [6], and Selvadurai [7, 8]). For example, the formal integral expressions [42] for the nonzero displacements components, $u_r(r,z)$ and $u_z(r,z)$, which were referred to the cylindrical polar coordinate system, take the forms

$$u_r(r,z) = \frac{P_B}{4\pi\mu}\int_0^\infty [(1-2v)-\xi z]\exp(-\xi z)J_1(\xi r)\,d\xi \tag{1}$$

$$u_z(r,z) = \frac{P_B}{4\pi\mu}\int_0^\infty [2(1-v)+\xi z]\exp(-\xi z)J_0(\xi r)\,d\xi \tag{2}$$

where J_0 and J_1 are, respectively, the zeroth-order and first-order Bessel functions of the first kind, μ is the linear elastic shear modulus and v is Poisson's ratio. Taking into consideration the direction of application of the Boussinesq force, these results can also be expressed in spherical polar coordinates (Selvadurai [7–9]) in the forms

$$2\mu u_R = \frac{P_B}{2\pi R}[4(1-v)\cos\Theta - (1-2v)] \tag{3}$$

$$2\mu u_\Theta = \frac{P_B \sin\Theta}{2\pi R}\left(-(3-4v) + \frac{(1-2v)}{(1+\cos\Theta)}\right) \tag{4}$$

where $r = R\sin\Theta$, $z = R\cos\Theta$ with $R\left(=\sqrt{r^2+z^2}\right)\in(0,\infty)$ and $\Theta\in(0,\pi/2)$. By appeal to the constitutive equations governing classical elasticity, the state of stress in the halfspace region can be determined uniquely from these results. As is evident, the displacements satisfy the regularity conditions necessary and sufficient to ensure uniqueness of the solution. Similarly, the stress state derived from (1) to (4) also satisfies the equations of equilibrium, the regularity conditions and the traction boundary conditions at the surface $z = 0$. Furthermore, the singularities in the stress state are integrable and contribute to a traction resultant identical to the force vector applied at the surface of the halfspace.

3. THE CONCENTRATION FACTOR

In this note, we focus attention on the result for the transfer of a concentrated normal force acting on the surface of a halfspace presented by Fröhlich [1], with specific reference to the exposition dealing with the transmission of stress within a halfspace region loaded at the surface by a concentrated normal force. The spatial decay of the stress as predicted by the classical theory of elasticity is found to be at variance with experimental observations of vertical stress distributions, in particular within geomaterial regions. The work of Fröhlich [1] is an empirical development, which adjusts the form of the vertical stress $\sigma_{zz}(r,z)$ because of Boussinesq's solution by introducing a 'concentration factor n', which allows the alteration of the decay pattern to suit an experimentally observed pattern. In the ensuing, we shall retain the presentation of Fröhlich [1] but replace the concentration factor v by n to avoid confusion with

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Poisson's ratio ν. It should be remarked at the outset that this semi-empirical modification is proposed *only* for incompressible ($\nu = 1/2$) elastic materials. (It should be noted that the '*limit of elastic incompressibility*' has rigorous mathematical and energetic bases because Poisson's ratio for isotropic elastic materials occupies the range $-1 \le \nu \le 1/2$ [9]. The constraint based on incompressibility is quite independent of the kinematic constraint based on compatibility of strains, which is a necessary and sufficient condition for the integrability of the strain–displacement relations to yield a unique displacement field.)

An inspection of the results by Fröhlich [1] indicates that the stress state reduces to a particularly simple form in the *limit of elastic incompressibility*. The mathematical basis for introducing this concept is lacking, in the sense that there appears to be no formal linear solution of the equations governing the classical theory of elasticity that will yield a solution to Boussinesq's problem for the action of a concentrated force P_B normal to the surface of a an elastic halfspace in the form

$$\sigma_{zz}(r,z) = -\frac{nP_B z^n}{2\pi R^{n+2}} \quad ; \quad R = \sqrt{r^2 + z^2} \tag{5}$$

The remaining stress components have been evaluated as follows:

$$\sigma_{rr}(r,z) = -\frac{nP_B r^2 z^{n-2}}{2\pi R^{n+2}} \quad ; \quad \sigma_{\theta\theta} = 0 \ ; \quad \sigma_{rz} = -\frac{nP_B r z^{n-1}}{2\pi R^{n+2}} \tag{6}$$

It can, however, be verified that, in the absence of body forces, the stress state defined by (5) and (6) also satisfies the nontrivial axisymmetric equations of equilibrium expressed in cylindrical polar coordinates:

$$\begin{aligned} &\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \\ &\frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \end{aligned} \tag{7}$$

and it can also be verified that on any plane $z = \text{const.}$,

$$\int_0^\infty \int_0^{2\pi} \sigma_{zz}(r,z) \ \ r\,dr\,d\theta + P_B = 0 \tag{8}$$

indicating that vertical equilibrium is satisfied at any plane $z = \text{const.}$ within the halfspace region. Similarly, it can be shown that on any cylindrical surface $r > 0$,

$$2\pi r \int_0^\infty \sigma_{rz}(r,z) \ \ dz + P_B = 0 \tag{9}$$

The results (8) and (9) are valid irrespective of the concentration factor n. Also, the classical Boussinesq's solution for $\sigma_{zz}(r,z)$ in an incompressible elastic medium is recovered when $n = 3$. Typical results for the distribution of the axial stress $|\sigma_{zz}(r,z)|$ for various values of n are shown in Figure 2. (Boussinesq's problem has no length parameter associated with it; consequently, a length parameter a is introduced to enable the presentation of the results in a nondimensional form.) Because the interpretation of the concentration factor-based analysis in terms of classical elasticity is valid only for $n = 3$; for any other choice of n, the solution should deviate from classical elastic behaviour, which satisfies only the equations of equilibrium but may not, *in general*, satisfy other equations applicable to classical elasticity. The objective of this note is to examine whether all governing equations of compatibility applicable to strains (Timoshenko and Goodier [40]; Davis and Selvadurai [6]; Selvadurai [8]) are satisfied to provide validity to the

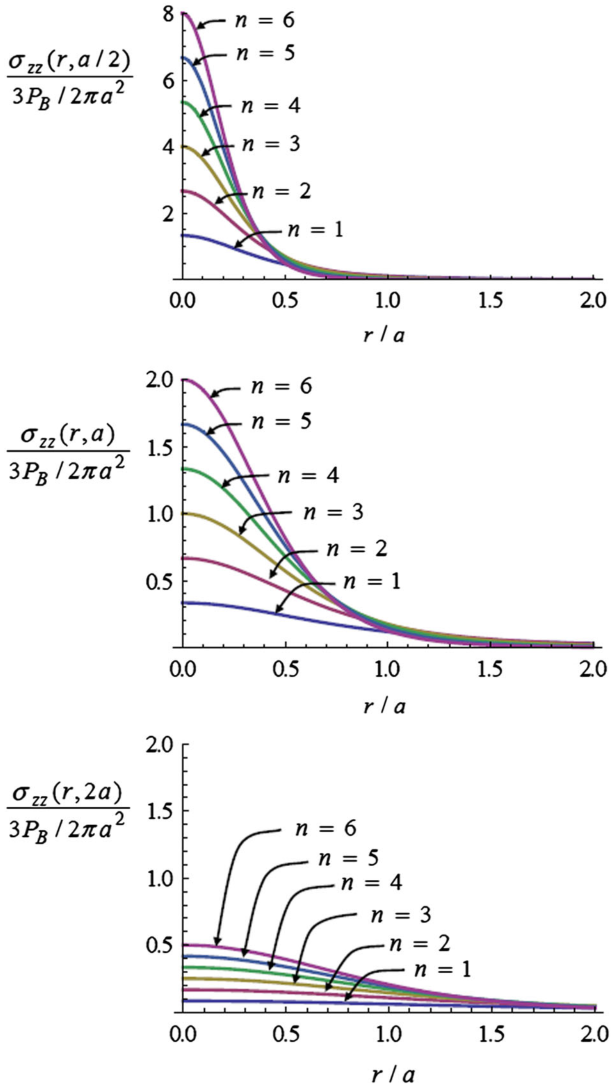

Figure 2. Influence of the concentration factor n on the distribution of axial stress $\sigma_{zz}(r,z)$.

continuum concept. In Cartesian notation, the Beltrami–Michell equations of compatibility applicable to a continuum region take the form

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_i} = \frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_l} + \frac{\partial^2 \varepsilon_{jl}}{\partial x_i \partial x_k} \tag{10}$$

Alternatively, in generalized tensor notation, the result (10) is equivalent to

$$\nabla \times \boldsymbol{\varepsilon} \times \nabla = 0 \tag{11}$$

where $\boldsymbol{\varepsilon}$ is the linearized strain tensor referred to the appropriate coordinate system, ∇ denotes the gradient operator and $\times$ denotes the cross product. A mathematical interpretation of the compatibility conditions relates to the vanishing of the Riemann-Christoffel tensor, which for infinitesimal strains gives rise to (11) and a physical interpretation is given in Figure 3, where the simply connectedness of the domain is preserved during the deformation.

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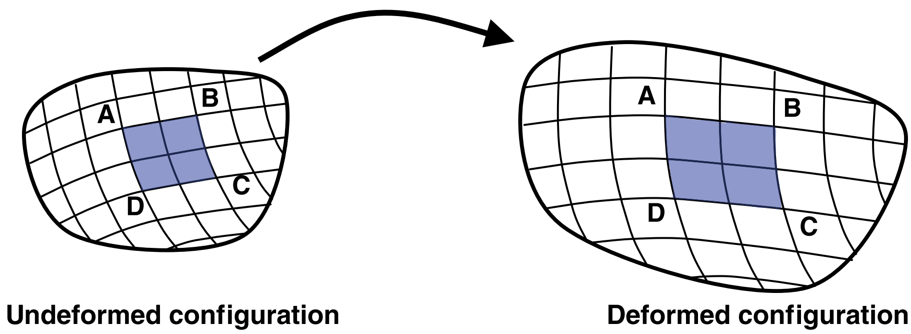

Figure 3. The concept of strain compatibility, which ensures the sequence of the positions of A,B,C and D.

Considering a cylindrical polar coordinate system (r,θ,z) and a state of axial symmetry characterized by the displacement field $\{u_r(r,z), 0, u_z(r,z)\}$, it can be shown that the nonzero components of the linearized strain tensor,

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{rr} & 0 & \varepsilon_{rz} \\ 0 & \varepsilon_{\theta\theta} & 0 \\ \varepsilon_{rz} & 0 & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & 0 & \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) \\ 0 & \dfrac{u_r}{r} & 0 \\ \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) & 0 & \dfrac{\partial u_z}{\partial z} \end{pmatrix} \tag{12}$$

should satisfy the four compatibility equations

$$\frac{\partial^2 \varepsilon_{\theta\theta}}{\partial z^2} - \frac{2}{r}\frac{\partial \varepsilon_{rz}}{\partial z} + \frac{1}{r}\frac{\partial \varepsilon_{zz}}{\partial r} = 0 \tag{13}$$

$$\frac{\partial^2 \varepsilon_{rr}}{\partial z^2} - 2\frac{\partial^2 \varepsilon_{rz}}{\partial r \partial z} + \frac{\partial^2 \varepsilon_{zz}}{\partial r^2} = 0 \tag{14}$$

$$\frac{\partial^2 \varepsilon_{\theta\theta}}{\partial r^2} + \frac{2}{r}\frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{1}{r}\frac{\partial \varepsilon_{rr}}{\partial r} = 0 \tag{15}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \varepsilon_{\theta\theta}}{\partial r} + \frac{1}{r}[\varepsilon_{\theta\theta} - \varepsilon_{rr}]\right) = 0 \tag{16}$$

Considering the stress state given by (5) and (6) and Hooke's law applicable to an *incompressible* isotropic elastic material, it can be shown that

$$\begin{aligned} \varepsilon_{rr} &= -\frac{nP_B}{4\pi E R^{n+2}}\left(2r^2 z^{n-2} - z^n\right) \quad ; \quad \varepsilon_{\theta\theta} = \frac{nP_B}{4\pi E R^{n+2}}\left(z^n + r^2 z^{n-2}\right) \\ \varepsilon_{zz} &= -\frac{nP_B}{4\pi E R^{n+2}}\left(2z^n - r^2 z^{n-2}\right) \quad ; \quad \varepsilon_{rz} = -\frac{nP_B}{4\pi E R^{n+2}}\left(3r z^{n-1}\right) \end{aligned} \tag{17}$$

It can be noted that the strain field (17) satisfies the incompressibility constraint

$$\mathrm{tr}\,\boldsymbol{\varepsilon} = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} \equiv 0 \tag{18}$$

for any choice of n. As noted previously, the incompressibility condition is obtained through a constitutive constraint rather than a kinematic constraint on the solution. Substituting (17) in (13) to

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(16), it can be shown that the compatibility equations will be satisfied *if and only if* $n \equiv 3$ and the compatibility equation is violated for all other values of n.

It would appear that Fröhlich's modification to Boussinesq's result to account for either *concentration* or *diffusion* of the load transmission pattern is a *statically admissible solution* but an incomplete result that violates the kinematics of deformation of a continuum. In hindsight, it is clear that *if* Fröhlich's solution for Boussinesq's concentrated force problem (which *completely satisfies* all the governing equations of classical elasticity, including equilibrium, compatibility and the constitutive equations of linear elasticity) is recovered *only* when $n \equiv 3$, then from consideration of Kirchhoff's uniqueness theorem in classical elasticity (Knops and Payne [43], Davis and Selvadurai [6], Selvadurai [9], Gurtin [44], Podio-Guidugli and Favata [45]) the solution must violate one or more of the governing equations when either $n < 3$ or $n > 3$. We have shown that the equations of equilibrium are satisfied for all choices of n; consequently, the compatibility equations must be violated for all choices of $n \neq 3$. The implications of violation of the compatibility conditions present itself in a nonuniqueness of the integration of the strain–displacement equations when determining the displacement components $u_r(r,z)$ and $u_z(r,z)$. For example, the displacement component $u_r(r,z)$ can be directly determined using the results for $\varepsilon_{\theta\theta}$ in (12) and the stress state (5) and (6);, that is,

$$u_r(r,z) = \frac{nP_B r \ z^{n-2}}{4\pi E R^n} \tag{19}$$

We can also obtain an expression for $u_r(r,z)$ by integrating the result in (12) for ε_{rr}, which gives

$$u_r(r,z) = \frac{P_B r \ z^{n-2}}{4\pi E R^n}\left(3 + \frac{(n-3)R^n}{z^n} \ {}_2\mathrm{F}_1{}^2\left[\frac{1}{2}, \frac{n}{2}, \frac{3}{2}; -\frac{r^2}{z^2}\right]\right) + G(z,n) \tag{20}$$

where ${}_2\mathrm{F}_1[a,b,c,d]$ is the hypergeometric function (Abramowitz and Stegun [46]) and (20) is indeterminate to within $G(z,n)$ an arbitrary function of z, which can be evaluated through an integration of the expressions for ε_{zz} and substituting the resulting expression for $u_z(r,z)$ and the expression (19) for $u_r(r,z)$ into the expression for ε_{rz}. For the present discussion, it is sufficient to consider the expressions (19) and (20); it is clear that the two expressions are distinctly different for $n \neq 3$, and when $n = 3$, both expressions reduce to

$$u_r(r,z) = \frac{3P_B r z}{4\pi E (r^2 + z^2)^{3/2}} \tag{21}$$

which is identical to Boussinesq's solution for the radial displacement in an incompressible elastic halfspace due to the action of the concentrated normal force. From (20), it is also abundantly clear that the arbitrary function $G(z,n)$ cannot produce a solution for $n \neq 3$, which will completely eliminate the second term within the brackets in (20) to yield the result (21).

Similarly, the result for ε_{zz} can be integrated with respect to z, which gives

$$u_z(r,z) = \frac{nP_B \ z^{n-1}}{4\pi E (n^2-1) r^n}\left(\begin{array}{c}\left(\left(n+1\right){}_2\mathrm{F}_1\left[\frac{n-1}{2}, \frac{n}{2}, \frac{(n+1)}{2}; -\frac{z^2}{r^2}\right]\right) \\ -3(n-1)\left(\frac{z^2}{r^2}\right) {}_2\mathrm{F}_1\left[1, \frac{3-n}{2}, \frac{3}{2}; -\frac{z^2}{r^2}\right]\end{array}\right) + F(r,n) \tag{22}$$

where $F(r,n)$ is an arbitrary function arising from the integration with respect to the variable z. In the specific case when $n = 3$, (22) reduces to

$$u_z(r,z) = \frac{3P_B}{4\pi E} \ \frac{r^2 + 2z^2}{(r^2+z^2)^{3/2}}\Bigg) - \frac{3P_B}{4\pi E}\left(\frac{1}{r}\right) + \widetilde{F}(r) \tag{23}$$

where $\widetilde{F}(r)$ depends on the variable r. The leading term on the right-hand side of (23) corresponds to Boussinesq's solution, and two additional terms are encountered in the reduction to the classical case.

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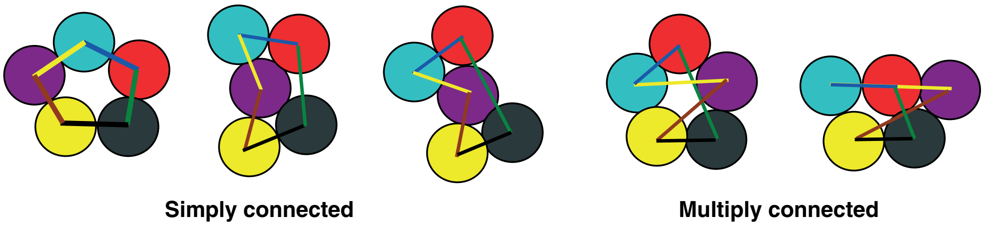

Figure 4. Loss of simply-connectedness in a particulate domain.

Without loss of generality, the arbitrary function $\widetilde{F}(r)$ can, however, be chosen such that Boussinesq's solution is recovered in the limit when $n = 3$. The detailed evaluation of the arbitrary functions $G(z,n)$ and $F(r,n)$ is possible, but these results are not central to the basic theme of the paper. It is sufficient to note that there is nonuniqueness in the evaluation of the displacement components from the integration of the strain–displacement relations, and this stems from the violation of the equations of compatibility, which are *necessary and sufficient* for the purposes of integration of the relevant equations.

4. CONCLUDING REMARKS

The concept of a concentration factor was introduced by Fröhlich, with the genuine intention of providing an approach that can account for the departure of observed results from predictions made using the results based on the classical theory of elasticity. In particular, the exact analytical solution for Boussinesq's problem for an isotropic elastic halfspace region is used to calibrate the 'concentration factor', n, which can alter the shape of either the *spreading* or the *concentration* of the load at depth. At the outset, it is clear, from Kirchhoff's uniqueness theorem in classical elasticity [6, 9, 43–45], that if Fröhlich's result converges to Boussinesq's result for $n \equiv 3$, the solution will not satisfy all the governing equations of elasticity when $n \neq 3$. The results presented in the paper indicate that Fröhlich's solution satisfies the equations of equilibrium, the boundary conditions and the stress–strain relations applicable to incompressible elastic materials but violates the equations of compatibility applicable to continua. This manifests in the form of a nonuniqueness in the displacement field obtained through the integration of the strain–displacement relations. Alternatively, if the kinematical relationships are not satisfied by Fröhlich's result, then the halfspace region must exhibit traits of a discontinuum similar to that of a particulate medium. It would also imply that the stress state associated with Fröhlich's solution is one that could be described by appeal to particulate mechanics similar to discrete element techniques, where the particle shape and reorientation can influence the stress transfer process. Figure 4 shows a typical configuration of a particulate medium where during deformation, an originally simply-connected configuration transforms to a multiply connected domain. Admittedly, substantial displacements of particles are needed to violate the simply-connectedness of an initial configuration.

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REFERENCES

1. Fröhlich OK. *Druckverteilung im Baugrunde. Mit Besonderer Berücksichtigung der Plastischen Erscheinungen*. Julius Springer: Wien, 1934.
2. Terzaghi K, Fröhlich OK. *Theorie der Setzung von Tonschichten*. F. Deuticke: Wien, 1936.
3. Fillunger P. *Erdbaumechanik?* Buchdruckerei Friedrich Jasper: Wien, 1936.
4. De Boer R. *Theory of Porous Media*. Springer-Verlag: Berlin, 2000.
5. Boussinesq J. *Application des Potentiels á l'Étude de l'Équilibre et du Mouvement des Solides Élastiques*. Gauthier-Villars: Paris, 1885.
6. Davis RO, Selvadurai APS. *Elasticity and Geomechanics*. Cambridge University Press: Cambridge, 1996.
7. Selvadurai APS. The analytical method in geomechanics. *Applied Mechanics Reviews* 2007; **60**:87–106.
8. Selvadurai APS. On Boussinesq's problem. *International Journal of Engineering Science* 2001; **39**:317–322.
9. Selvadurai APS. *Partial Differential Equations in Mechanics. Vol. 2. The Biharmonic Equation, Poisson's Equation*. Springer-Verlag: Berlin, 2000.
10. Mindlin RD. *Force at a point in the interior of a semi-infinite solid Physics* 1936; **7**:195–202.
11. Thompson W. (Lord Kelvin) On the equations of equilibrium of an elastic solid. *The Cambridge and Dublin Mathematical Journal* 1848; **3**:87–89.
12. Love AEH. *A Treatise on the Mathematical Theory of Elasticity*. Cambridge University Press: Cambridge, 1927.
13. Westergaard HM. *Elasticity and Plasticity*. Harvard University Press: Mass, 1952.
14. Gladwell GML. *Contact Problems in the Classical Theory of Elasticity*. Sijthoff and Noordhoff: The Netherlands, 1980.
15. Mura T. *Micromechanics of Defects in Solids*. Martinus Nijhoff Publishers: Dordrecht, 1987.
16. Ting TCT. *Anisotropic Elasticity*. Oxford University Press: Oxford, 1996.
17. Hetnarski RB, Ignaczak J. *Mathematical Theory of Elasticity*. Taylor and Francis: New York, 2004.
18. Griffith JH. The pressure under substructures. *Engineering Contracting* 1929; **68**:113–119.
19. Soehne W. Fundamentals of pressure distribution and soil compaction under tractor tires, *Agricultural Engineering* 1958; **39**:276–281, 290.
20. Dexter AR, Horn R, Holloway R, Jakobsen BF. Pressure transmission beneath wheels in soils on the Eyre Peninsula of South Australia. *Journal of Terramechanics* 1988; **25**:135–147.
21. Ohde J. Zur Theorie der Druckverteilung im Baugrund. *Bauingenieur* 1939:451.
22. Borowicka H. Die druckausbreitung im halbraum bei linear zunehmendem elastizitätsmodul. *Ingenieur-Archive* 1942; **14**:75–82.
23. Gibson RE. Some results concerning displacements and stresses in a non-homogeneous elastic half-space. *Geotechnique* 1967; **17**:58–67.
24. Selvadurai APS, Singh BM, Vrbik J. A Reissner–Sagoci problem for a non-homogeneous elastic solid. *Journal of Elasticity* 1986; **16**:383–391.
25. Selvadurai APS. The settlement of a rigid circular foundation resting on a half-space exhibiting a near surface elastic non-homogeneity. *International Journal for Numerical and Analytical Methods in Geomechanics* 1996; **20**:351–364.
26. Selvadurai APS, Katebi A. Mindlin's problem for an incompressible elastic half-space with an exponential variation in the linear elastic shear modulus. *International Journal of Engineering Science* 2013; **65**:9–21.
27. Katebi A, Selvadurai APS. Undrained behaviour of a non-homogeneous elastic medium: the influence of variations in the elastic shear modulus with depth. *Geotechnique* 2013; doi.org/10.1680/geot.12.P.164.
28. Terzaghi K. *Theoretical Soil Mechanics*. John Wiley and Sons: New York, 1943.
29. Tschebotarioff GP. *Soil Mechanics, Foundations and Earth Structures*. McGraw-Hill Book Company Inc.: New York, 1951.
30. Jumikis AR. *Mechanics of Soils. Fundamentals for Advanced Study*. D. Van Nostrand Co. Inc: Princeton, New Jersey, 1964.
31. Széchy K, Varga L. *Foundation Engineering. Soil Exploration and Spread Foundations*. Akadémiai Kiadó: Budapest, 1978.
32. Klein GK. On the equations proposed by OK Fröhlich, *Vestnik Inzhenerov i Tekhnikov* 1948; 2.
33. Korenev BG. Structures resting on an elastic foundation, Ch. III. In *Structural Mechanics in the USSR 1917–1957*, Rabinovich IM (ed.). Pergamon Press, Oxford, 1960; 160–190.
34. Keller T, Lamandé M, Peth S, Berli M, Delenne J-Y, Baumgarten W, Rabbel W, Radjai F, Rajchenbach J, Selvadurai APS, Or D. An interdisciplinary approach towards improved understanding of soil deformation during compaction. *Soil and Tillage Research* 2013; **128**:61–80.
35. Hast N. Measuring stresses and deformations in soil materials. Ingenjo¨ rsvetenskapsakad, Handl. No. 178, Stockholm, Sweden, 1945.
36. Hvorslev MJ. The changeable interaction between soils and pressure cells: tests and reviews at the waterways experiment station. Technical Report No. 5-76-7, U.S. Army Corps of Engineers, Waterways Experiment Station, Vicksburg, MI, 1976.
37. Selvadurai APS. *Elastic Analysis of Soil-Foundation Interaction, Developments in Geotechnical Engineering, 17*. Elsevier Sci Publ: The Netherlands, 1979.
38. Hanna TH. *Field Instrumentation in Geotechnical Engineering*. Trans-Tech. Publ.: Clausthal-Zellerfeld, 1985.
39. Selvadurai APS, Labanieh S, Boulon MJ. On the mechanics of contact between a flexible transducer diaphragm located at a rigid boundary and an elastic material. *International Journal for Numerical and Analytical Methods in Geomechanics* 2007; **31**:933–952.

DOI: 10.1002/nag

40. Timoshenko SP, Goodier JN. *Theory of Elasticity*. McGraw-Hill: New York, 1970.
41. Little RW. *Elasticity*. Prentice-Hall: Upper Saddle River, New York, 1973.
42. Sneddon IN, Berry DS. The classical theory of elasticity, *Handbuch der Physik*, Flugge S (ed.). Springer-Verlag, Berlin 1958; **VI**:1–123.
43. Gurtin ME. ‘The linear theory of elasticity’, in *Mechanics of Solids II*, *Encyclopaedia of Physics* Flugge S (ed.). Springer-Verlag, Berlin, 1972; **VIa/2**:1–295.
44. Knops RJ, Payne LE. *Uniqueness Theorems in Linear Elasticity*. Springer-Verlag: Berlin, 1971.
45. Podio-Guidugli P, Favata A. *Elasticity for Geotechnicians*. Springer Int Publ: Switzerland, 2014.
46. Abramowitz M, Stegun IA. *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. Dover: New York, 1964.

DOI: 10.1002/nag